Applications of the Gauge Principle to Gravitational Interactions

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Abstract

The idea of applying the gauge principle to formulate the general theory of relativity started with Utiyama in 1956. I review various applications of the gauge principle applied to different aspects of the gravitational interactions.

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Contents

1   Introduction

Gauge invariance now plays a fundamental role in theoretical physics. In order to have a Lorentz invariant formulation of electromagnetism, one is forced to represent the two polarizations of the massless photon with a four-component vector. The redundancy present in the additional components of the four-vector is then reflected by the fact that the Lagrangian is invariant under an additional gauge transformation that allows to fix one of the space components of the vector A_i, the other component A_0 being non dynamcial. In general relativity the problem is more difficult as one must represent the two polarizations of the massless graviton by a symmetric four-dimensional tensor with ten independent components. What makes this possible is the diffeomorphism invariance of the action which allows to fix four of the ten parameters, the other four components g_{0i} being non-dynamical [1]. This, however, is not the unique way to represent the gravitational field. The

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other possibility, originally due to Weyl [2] and Cartan [3], and in a more concrete form to Utiyama [4] and Kibble [5], is based on representing the two polarizations of the graviton by the vierbein e^a_μ with sixteen independent components which transform as a four-vector under diffeomorphisms and a Lorentz vector under Lorentz transformations. The action is constructed to be invariant under both diffeomorphism and local Lorentz transformations. The gauge field associated with local Lorentz invariance, $\omega_\mu{}^{ab}$ is not taken to be an independent field, but rather determined by setting the generalized gauge covariant derivative of the field e^a_μ, with respect to both symmetries, to vanish [6]

$$\mathcal{D}_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega_\mu{}^{ab} e^b_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho = 0,$$

where $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$ is a symmetric connection. This system of 64 independent equations allows to solve uniquely for the 24 components of the spin-connection $\omega_\mu{}^{ab}$ and for the 40 independent components $\Gamma^\rho_{\mu\nu}$ in function of e^a_μ provided that e^a_μ is invertible

$$\omega_{\mu ab}(e) = \frac{1}{2} e^\nu_a e^\rho_b \left(\Omega_{\mu\nu\rho}(e) - \Omega_{\nu\rho\mu}(e) + \Omega_{\rho\mu\nu}(e) \right),$$

$$\Omega_{\mu\nu\rho} = \left(\partial_\mu e^c_\nu - \partial_\nu e^c_\mu \right) e_{\rho c},$$

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma} \right),$$

$$g_{\mu\nu} = e^a_\mu e_{\nu a},$$

which shows that the connection $\Gamma^\rho_{\mu\nu}$ is identical to the Christoffel connection. Taking the antisymmetrized derivative of the metric condition gives an identity between the curvature of the spin-connection and the curvature of the Christoffel connection

$$0 = R_{\mu\nu}{}^{ab}(\omega) e_{\rho b} - R^\sigma{}_{\rho\mu\nu}(\Gamma) e^a_\sigma,$$

$$R_{\mu\nu}{}^{ab}(\omega) = \partial_\mu \omega_\nu{}^{ab} - \partial_\nu \omega_\mu{}^{ab} + \omega_\mu{}^{ac} \omega_{\nu c}{}^b - \omega_\nu{}^{ac} \omega_{\mu c}{}^b,$$

$$R^\sigma{}_{\rho\mu\nu}(\Gamma) = \partial_\mu \Gamma^\sigma_{\nu\rho} - \partial_\nu \Gamma^\sigma_{\mu\rho} - \Gamma^\lambda_{\mu\rho} \Gamma^\sigma_{\nu\lambda} + \Gamma^\lambda_{\nu\rho} \Gamma^\sigma_{\mu\lambda},$$

This identity implies that an invariant Lagrangian can be formed out of either curvature

$$e^\mu_a e^\nu_b R_{\mu\nu}{}^{ab}(\omega) = g^{\rho\nu} R^\mu{}_{\rho\mu\nu}(\Gamma) \equiv R.$$

In this way, the Lorentz and diffeomorphism invariant Lagrangian becomes either a function of e^a_μ only or a function of $g_{\mu\nu}$ only. The equivalence of both

expressions can be seen by noting that the Lorentz gauge transformations can be used to fix the six antisymmetric components of e_μ^a to vanish. Then in both cases diffeomorphism invariance fixes four more components out of the six $g_{ij}, i, j = 1, 2, 3$, with the four components g_{0i} being non dynamical, leaving only two dynamical degrees of freedom, as should be the case. This shows the equivalence of the vierbein and metric formulations of the general theory of relativity, a result which has been dealt with intensively in the literature. This classical equivalence was culminated in the work of Ashtekar [7], and collaborators where it was shown that the $SL(2, \mathbb{C})$ invariance can be written in such a way as to have a manifest invariance under the complex group $SU(2)$ [8] so that the gauge fields could be separated into self-dual and anti self-dual parts. The main advantage of the Ashtekar formulation is that it allows to take the spin-connection as the canonical variables while the inverse of the soldering forms e_a^μ are taken as the conjugate variables, which allowed to formulate a theory for loop quantum gravity [9].

The requirement that the Lagrangian should have local $SL(2, \mathbb{C})$ invariance is powerful [10] and fits with the fact that space-time spinors do exist in nature. The Dirac equation in Minkowski space-time has global $SL(2, \mathbb{C})$ invariance, and it is natural to require that this invariance be promoted to become local by introducing the spin-connection as a gauge field. There is, however, a need to introduce the vierbein, or soldering form as an external field. Naturally, the idea of extending the homogeneous Lorentz invariance to the inhomegeneous Lorentz invariance was exploited [11], [12], [13]. In this case the group has translation generators in addition to the rotation generators. The field strengths associated with the translation generators are constrained to be zero, allowing the identification of the translation gauge parameters with the diffeomorphisms parameters. This is where a gauge theory of gravity differs from the usual Yang-Mills type gauge theory, as the constraint of vanishing translational field strength allows to solve for the spin connection as function of the vierbein, which in this case is the gauge field associated with translations. This makes the field strength associated with rotations depend on second derivatives of the vierbein, and becomes identified with the curvature of the metric formed from the vierbein, as shown above. The constraints render the theory non-renormalizable. This explains why the method of formulating gravity as a gauge theory of the inhomogeneous Lorentz group does not lead to improvements in the renormalizability of the theory. The main advantages lie in the simplicity of formulation, in having a polynomial structure, and in the prospect of extending the gravitational

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theory to be unified with the other interactions.

The gauge principle has played a prominent role in the formulation of different aspects of gravitational theories. This was one of the guiding principles of my research for the last thirty years. In this article I will review the various ways of formulating theories of gravity based on the gauge principle. The recent interest in models of bigravity resulting from brane models makes it necessary to study the mechanism of generating a small mass to one combination of the two metrics in a consistent way. For some time it was thought that a theory for a massive graviton is inconsistent because it does not have a smooth limit to the massless case [14],[15],[16]. We shall show that, in analogy with gauge theory for spin-1 fields, it is possible to have a consistent theory by generating the mass through spontaneous symmetry breaking and the use of the Higgs mechanism. I will also study the possibility of extending the general theory of relativity to describe a complex Hermitian metric, an idea first considered by Einstein in 1945 [17],[18], in his attempt to unify gravity with electromagnetism. It is now known that the antisymmetric part of the metric does not describe the electromagnetic field but rather an antisymmetric tensor [19], [20] , which for consistency at the non-linear level, should be massive [21]. The other possibility to be explored is the unification of internal symmetries and space-time symmetries. This corresponds to a theory of a complex $U(N)$ valued metric [22], [23]. We shall also show that it is possible to construct simple supergravity [24] using the gauge method by considering graded algebras [25], [26],[27], [28], [29]. Finally, we show that when space-time coordinates do not commute, which makes it necessary to replace ordinary products with Moyal products [30], it is possible to deform Einstein's gravity using the methods learned from the gauge formulations of gravity. The plan of this paper is as follows. In section two we briefly review the formulation of gravity as a gauge theory of the inhomogeneous Lorentz group. In section three we consider a gauge theory with two gravitons and show how to construct a consistent theory of one massless and one massive graviton by employing the Higgs mechanism. In section four we consider a gauge theory of gravity based on gauging the unitary group $U(1,3)$ and show that this is closely related to general relativity with Hermitian metric. In section five we consider unification of space-time and internal symmetries, and show that it is possible to construct a gauge theory of gravity with matrix valued metrics, where only one graviton remains massless with the other gravitons acquiring mass through the spontaneous breakdown of the larger symmetry. In section six we briefly discuss gauging graded algebras to con-

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struct simple supergravity. In section seven we consider topological gauge theories of gravity based on Chern-Simons forms with the gauge groups taken to be generalizations of the rotation groups in higher dimensions. In section eight we consider the noncommutative extension of the gauge formulation of gravity where usual products are replaced with Moyal products. Section nine is the conclusion.

2 Gauging the inhomogeneous Lorentz group

It is natural to generalize global invariance of the Dirac Lagrangian to a local one under Lorentz transformations

$$\delta\psi = \Omega\psi,$$

where $\Omega = \exp\left(\frac{1}{4}\lambda^{ab}\gamma_{ab}\right)$ and $\gamma_{ab} = \frac{1}{2}\left[\gamma_a, \gamma_b\right]$ by allowing the gauge parameters λ^{ab} to depend on x^μ. The invariance is achieved by replacing the ordinary derivative ∂_μ with the covariant derivative [4], [5]

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{\ ab}\gamma_{ab}.$$

Utiyama proposed to realize gravity as a gauge theory of the homogeneous Lorentz group with local gauge invariance under space-time coordinate rotations. This step was then generalized to the inhomegeneous Lorentz group to enforce invariance under local translations as well. The connection associated with this group is given by [11], [12],[13],[31]

$$\nabla_\mu = \partial_\mu + e_\mu^a P_a + \omega_\mu^{\ ab}J_{ab},$$

where P_a are the group generators associated with translation and J_{ab} are the rotation generators, with e_μ^a and $\omega_\mu^{\ ab}$ being the respective gauge fields. The curvature associated with this connection is evaluated to give

$$\left[\nabla_\mu, \nabla_\nu\right] = T_{\mu\nu}^{\ \ a}P_a + R_{\mu\nu}^{\ \ ab}J_{ab},$$

where

$$T_{\mu\nu}^{\ \ a} = \partial_\mu e_\nu^a + \omega_\mu^{\ ab}e_{\nu b} - \mu \leftrightarrow \nu,$$
$$R_{\mu\nu}^{\ \ ab} = \partial_\mu \omega_\nu^{ab} + \omega_\mu^{\ ac}\omega_{\nu c}^{\ \ b} - \mu \leftrightarrow \nu.$$

The gauge transformation of the vierbein e_μ^a is given by

$$\delta e_\mu^a = \partial_\mu \zeta^a + \omega_\mu{}^{ab}\zeta_b + \lambda^{ab}e_{\mu b}.$$

Setting the torsion $T_{\mu\nu}{}^a$ to zero allows to solve for ω_μ^{ab} uniquely provided that the field e_μ^a is invertible. This is the same equation as that obtained by antisymmetrizing the metric condition

$$\mathcal{D}_\mu e_\nu^a - \mathcal{D}_\nu e_\mu^a = T_{\mu\nu}{}^a = 0.$$

The number of independent components in $T_{\mu\nu}{}^a$ matches the number of independent components in ω_μ^{ab}. The presence of this constraint is the main difference between gravity and Yang-Mills gauge theories. This constraint breaks the translational gauge invariance. We note, however, that we can write for the gauge transformation of e_μ^a,

$$\begin{aligned}
\delta e_\mu^a &= \partial_\mu \zeta^\nu e_\nu^a + \zeta^\nu \partial_\mu e_\nu^a + \omega_\mu{}^{ab}\zeta_b + \lambda^{ab}e_{\mu b} \\
&= \partial_\mu \zeta^\nu e_\nu^a + \zeta^\nu \left(\partial_\nu e_\mu^a + \omega_\nu{}^{ab}e_{\mu b} - \omega_\mu{}^{ab}e_{\nu b} + T_{\mu\nu}{}^a\right) + \omega_\mu{}^{ab}\zeta_b + \lambda^{ab}e_{\mu b} \\
&= \partial_\mu \zeta^\nu e_\nu^a + \zeta^\nu \partial_\nu e_\mu^a + \lambda'^{ab}e_{\mu b} + \zeta^\nu T_{\mu\nu}{}^a,
\end{aligned}$$

where $\zeta^\nu = \zeta^a e_a^\nu$ becomes the parameter for general coordinate transformations provided that the torsion vanishes. One can require that the transformations of $\omega_\mu{}^{ab}$ be modified so as to preserve the zero torsion constraint. The action is constructed in terms of the gauge covariant curvature $R_{\mu\nu}^{ab}$ and the gauge fields e_μ^a

$$I = \int d^4x \epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}e_\mu^a e_\nu^b R_{\kappa\lambda}{}^{cd}(\omega).$$

This is invariant under Lorentz rotations and diffeomorphisms. It is also invariant under translation invariance provided the torsion is set to zero:

$$\begin{aligned}
\delta I &= 2\int d^4x \epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}\nabla_\mu \zeta^a e_\nu^b R_{\kappa\lambda}{}^{cd}(\omega) \\
&= -\int d^4x \epsilon^{\mu\nu\kappa\lambda}\epsilon_{abcd}\zeta^a T_{\mu\nu}^b R_{\kappa\lambda}{}^{cd}(\omega) \\
&= 0,
\end{aligned}$$

where we have integrated by parts and used the Bianchi identity $\nabla_{[\mu}R_{\kappa\lambda]}{}^{cd}(\omega) = 0$. Therefore enforcing the zero torsion constraint allows to consider an invariant action, without the need to define a metric on the four-dimensional

manifold, by making use of the differential form representation. Thus, let
[32]

$$e^a = e^a_\mu dx^\mu, \quad \omega^{ab} = \omega^{ab}_\mu dx^\mu,$$

and

$$R^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b \equiv \frac{1}{2} R_{\mu\nu}{}^{ab} dx^\mu \wedge dx^\nu,$$

so that the invariant action simplifies to

$$I = \int_M \epsilon_{abcd} e^a \wedge e^b \wedge R^{cd}.$$

To this one can always add a cosmological constant

$$\int_M \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d.$$

One can further write the above action in an index free notation be utilizing
the $SL(2, \mathbb{C})$ invariance [10]. Thus, let

$$e = e^a \gamma_a,$$

$$\omega = \frac{1}{4} \omega^{ab} \gamma_{ab},$$

which transform as

$$e \rightarrow \Omega^{-1} e \, \Omega, \quad \omega \rightarrow \Omega^{-1} \omega \, \Omega + \Omega^{-1} d \, \Omega,$$

under $SL(2, \mathbb{C})$ transformations. The action then simplifies to

$$I = 2 \int_M Tr \left(i\gamma_5 e \wedge e \wedge R \right),$$

where

$$R = d\omega + \omega \wedge \omega,$$

and invariance of the action follows from the commutativity of γ_5 with Ω
[33].

3 Massive gravity through spontaneous symmetry breaking

In the Kaluza-Klein approach of gravity in higher dimensions, the components of the higher dimensional metric tensor along the four-dimensional subspace is expanded in terms of Fourier components

$$g_{\mu\nu}(x,y) = \prod_{i=1}^{D-4}\sum_{n_i=0}^{\infty} g_{\mu\nu n_1\cdots n_{D-4}}(x) e^{in_i y^i}$$

where y^i are the coordinates of the compact directions. Besides the zero mode representing the massless graviton, one gets an infinite number of massive gravitons whose masses which are multiples of the Planck mass. In brane models of gravity as well in gravitational models in noncommutative geometry it is possible to get gravitons with small mass [34]. The linearized Lagrangian for massive spin-2 field $h_{\mu\nu}$ was found by Fierz and Pauli to be given by [35]

$$I = -\frac{1}{4}\int d^4x \left(\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} - 2\partial^\nu h_{\mu\nu}\partial_\lambda h^{\mu\lambda} + 2\partial^\nu h_{\mu\nu}\partial^\mu h_\lambda^\lambda - \partial_\mu h_\nu^\nu \partial^\mu h_\lambda^\lambda \right.$$
$$\left. + m^2\left(h_{\mu\nu}h^{\mu\nu} - b h_\nu^\nu h_\lambda^\lambda \right)\right),$$

where, for consistency b must be set to 1 so as to guarantee that only five components of $h_{\mu\nu}$ propagate, instead of the expected six. The mass independent part of the above Lagrangian is the same as the one obtained by linearizing the Einstein-Hilbert action around a Minkowski background. The propagator for $h_{\mu\nu}$ is [36]

$$\Delta_{\mu\nu}^{\rho\sigma} = \frac{1}{m^2 - k^2}\left(\left(\delta_\mu^\rho - \frac{k_\mu k^\rho}{m^2}\right)\left(\delta_\nu^\sigma - \frac{k_\nu k^\sigma}{m^2}\right)\right.$$
$$- \frac{1}{3}\left(\eta_{\mu\nu} - \frac{k_\mu k_\nu}{m^2}\right)\left(\eta^{\rho\sigma} - \frac{k^\rho k^\sigma}{m^2}\right)$$
$$\left. + \frac{1-b}{2(1-b)k^2 + (1-4b)m^2}\left(\eta_{\mu\nu} + \frac{2k_\mu k_\nu}{m^2}\right)\left(\eta^{\rho\sigma} + \frac{2k^\rho k^\sigma}{m^2}\right)\right).$$

Notice that when $b \neq 1$, the massive spin-2 field and the ghost of the spin-0 are coupled, and will only decouple for $b = 1$, which is the Fierz-Pauli choice. The choice $b = 1$ cannot be maintained at the quantum level and has to be

tuned. The symmetric tensor $h_{\mu\nu}$ has ten independent components but only the six components h_{ij} $(i, j = 1, 2, 3)$ are dynamical. A massive spin-2 field must have only five dynamical degrees of freedom $(2j + 1 = 5)$. This implies that there is an additional component, a spin-0 ghost, that does not decouple except for the choice $b = 1$. The limit of this propagator to the massless case $m \to 0$ is singular and is similar to the propagator of a massive spin-1 field, which is also singular in the massless limit. This strongly suggests that in order to solve the problem of the singular zero mass limit, the mass of the spin-2 field should be acquired through spontaneous symmetry breaking and the Higgs mechanism. To achieve this, the Lagrangian must have a gauge symmetry to be broken. This makes it necessary to extend the symmetry of the system, but in such a way as not to increase the dynamical degrees of freedom of the system. This is where the gauge principle of formulating gravity enters.

To illustrate the mechanism, we consider a coupled system of one massless graviton and one massive graviton formulated as a gauge theory of $SP(4) \times SP(4)$ [37]. We start with the gauge fields

$$A_{\mu\alpha}{}^{\beta} = \left(ie_{\mu}^{a}\gamma_{a} + \frac{1}{4}\omega_{\mu}^{ab}\gamma_{ab} \right)_{\alpha}^{\beta},$$

$$A'_{\mu\alpha}{}^{\beta} = \left(ie_{\mu}'^{a}\gamma_{a} + \frac{1}{4}\omega_{\mu}'^{ab}\gamma_{ab} \right)_{\alpha}^{\beta},$$

where $(A_{\mu}C)_{\alpha\beta} = (A_{\mu}C)_{\beta\alpha}$, C being the charge conjugation matrix. These also satisfy the reality conditions

$$\gamma_0 A_{\mu}^{\dagger}\gamma_0 = -A_{\mu}, \quad \gamma_0 A_{\mu}'^{\dagger}\gamma_0 = -A_{\mu}'.$$

These have the following gauge transformations

$$A_{\mu} \to \Omega A_{\mu}\Omega^{-1} + \Omega\partial_{\mu}\Omega^{-1}$$
$$A_{\mu}' \to \Omega' A_{\mu}'\Omega'^{-1} + \Omega'\partial_{\mu}\Omega'^{-1}$$

where Ω and Ω' denote independent symplectic matrices as gauge parameters

$$\Omega = \exp\left(i\lambda^{a}\gamma_{a} + \frac{1}{4}\lambda^{ab}\gamma_{ab} \right),$$

$$\Omega' = \exp\left(i\lambda'^{a}\gamma_{a} + \frac{1}{4}\lambda'^{ab}\gamma_{ab} \right),$$

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satisfying $\Omega^{-1} = C\Omega^T C^{-1}$. Notice that the field e_μ^a is now associated with translation in the internal space $SP(4)$. In other words, the $SL(2,\mathbb{C})$ symmetry is now extended to $SP(4)$. Next, introduce a Higgs fields G subject to the reality condition

$$\gamma_0 G^\dagger \gamma_0 = CG^T C^{-1} \equiv \widetilde{G}$$

transforming under the product representation of $SP(4) \times SP(4)$

$$G \to \Omega G \Omega'^{-1},$$

which has the following expansion in the Clifford algebra basis

$$G_\alpha^{\beta'} = \left(\varphi + i\pi\gamma_5 + iv^a\gamma_5\gamma_a + ig^a\gamma_a + g^{ab}\gamma_{ab}\right)_\alpha^{\beta'}.$$

For the theory to possess a stable Poincaré invariant vacuum solution, we require G to have a non-vanishing vacuum:

$$\left\langle G_\alpha^{\beta'} \right\rangle = (a + ib\gamma_5)_\alpha^{\beta'}.$$

This breaks the symmetry spontaneously from $SP(4) \times SP(4)$ to $SL(2,\mathbb{C})$ through a non-linear realization [38]. The number of independent components of G needed to parametrize the homogeneous space

$$\frac{SP(4) \times SP(4)}{SL(2,\mathbb{C})},$$

is $10 + 10 - 6 = 14$. Thus two constraints must be imposed on G to reduce the number of independent components from 16 to 14. For example these can be taken to be

$$Tr\left(G\widetilde{G}\right) = 4c_1,$$
$$Tr\left(\left(G\widetilde{G}\right)^2\right) = 4c_2,$$

where $\widetilde{G} = CG^T C^{-1} \to \Omega'\widetilde{G}\Omega^{-1}$. The action is taken to be of the form

$$\int Tr\left(\alpha G\widetilde{G}F \wedge F + \alpha'\widetilde{G}GF' \wedge F'\right.$$
$$\left. +\beta\nabla G \wedge \nabla\widetilde{G} \wedge \nabla G \wedge \nabla\widetilde{G} + \beta'\nabla G \wedge \nabla\widetilde{G} \wedge \nabla G \wedge \nabla\widetilde{G}\right),$$

where

$$F = dA + A \wedge A,$$
$$F' = dA' + A' \wedge A',$$
$$\nabla G = dG + AG - GA',$$
$$\nabla \widetilde{G} = d\widetilde{G} + A'\widetilde{G} - \widetilde{G}A,$$

To analyze the physical content of the Lagrangian we can eliminate the 14 remaining components of G by fixing 14 gauge conditions, with the remaining gauge freedom corresponding to the unbroken local $SL(2, \mathbb{C})$ invariance. To derive the component form of the Lagrangian in this unitary gauge, we write

$$F = \frac{1}{2} \left(iF_{\mu\nu}{}^{a}\gamma_a + F_{\mu\nu}{}^{ab}\gamma_{ab} \right) dx^{\mu} \wedge dx^{\nu}$$

$$F' = \frac{1}{2} \left(iF'_{\mu\nu}{}^{a}\gamma_a + F'_{\mu\nu}{}^{ab}\gamma_{ab} \right) dx^{\mu} \wedge dx^{\nu}$$

$$F_{\mu\nu}{}^{a} = T_{\mu\nu}{}^{a}, \qquad F'_{\mu\nu}{}^{a} = T'_{\mu\nu}{}^{a}$$

$$F_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} - 4 \left(e^a_{\mu} e^b_{\nu} - e^a_{\nu} e^b_{\mu} \right), \qquad F'_{\mu\nu}{}^{ab} = R'_{\mu\nu}{}^{ab} - 4 \left(e'^a_{\mu} e'^b_{\nu} - e'^a_{\nu} e'^b_{\mu} \right)$$

$$\nabla_{\mu} G = \left(i \left(a e^{a-}_{\mu} - i b e^{a+}_{\mu} \gamma_5 \right) \gamma_a + \frac{1}{4} \omega_{\mu}{}^{-ab} \left(a + i b \gamma_5 \right) \gamma_{ab} \right)$$

$$\nabla_{\mu} \widetilde{G} = \left(-i \left(a e^{a-}_{\mu} + i b e^{a+}_{\mu} \gamma_5 \right) \gamma_a - \frac{1}{4} \omega_{\mu}{}^{-ab} \left(a + i b \gamma_5 \right) \gamma_{ab} \right)$$

where

$$e^{a\pm}_{\mu} = e^a_{\mu} \pm e'^a_{\mu},$$
$$\omega_{\mu}{}^{-ab} = \omega_{\mu}{}^{ab} - \omega'_{\mu}{}^{ab}.$$

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One then finds that the action simplifies to [37]

$$
I = \int_M d^4x \epsilon^{\mu\nu\rho\sigma} \left(\alpha_1 F_{\mu\nu ab} F_{\rho\sigma}{}^{ab} + \alpha_1' F'_{\mu\nu ab} F'_{\rho\sigma}{}^{ab} \right)
$$

$$
+ \epsilon_{abcd} \int_M d^4x \epsilon^{\mu\nu\rho\sigma} \left(\beta_1 F_{\mu\nu}{}^{ab} F_{\rho\sigma}{}^{cd} + \beta_1' F'_{\mu\nu}{}^{ab} F'_{\rho\sigma}{}^{cd} \right)
$$

$$
+ \epsilon_{abcd} \int_M d^4x \epsilon^{\mu\nu\rho\sigma} \left(\gamma_1 e_\mu^{a-} e_\nu^{b-} e_\rho^{c-} e_\sigma^{d+} + \gamma_1' e_\mu^{a+} e_\nu^{b+} e_\rho^{c+} e_\sigma^{d-} \right)
$$

$$
+ \epsilon_{abcd} \int_M d^4x \epsilon^{\mu\nu\rho\sigma} \left(\delta_1 \left(e_\mu^{a+} e_{\nu e}^{-} - e_\mu^{a-} e_{\nu e}^{e} \right) \omega_\rho^{-bc} \omega_\sigma^{-de} + \delta_1' e_\mu^{a+} e_\nu^{b-} \omega_\rho^{-be} \omega_{\sigma e}^{-d} \right),
$$

where the coefficients appearing above depend on $\alpha, \alpha', \beta, \beta', \gamma, \gamma', a$ and b. Notice that no metric is needed to define this action and all terms appearing correspond to four-forms. It contains topological terms corresponding to Euler and Gauss-Bonnet invariants, as well as kinetic terms for the two metrics e_μ^{a+} and e_μ^{a-}. The physical spectrum of a similar Lagrangian was carried a long time ago [37], where it was shown that one obtains one combination of the two vierbeins to represent a massless graviton with the other combination representing a massive graviton whose mass can be adjusted to take very small values. The linearized form of the above Lagrangian is of the Pauli-Fierz type where there are two dynamical degrees of freedom for the massless graviton and five degrees for the massive graviton. As mentioned earlier, the propagator for the massive graviton is singular in the zero mass limit. It is then necessary to study such a limit in a non-unitary gauge, where the extended gauge invariance is manifest, and where the components of the Higgs field G are not gauge fixed. For example one can partially fix G to take the form

$$
G = a + i\pi\gamma_5 + iv^a\gamma_5\gamma_a.
$$

In this case one can show that the physical degrees of freedom correspond to each of the two massless spin-2 polarizations for e_μ^{a+} and e_μ^{a-} as well as two degrees for the transverse polarizations in v^a and one degree for the spin-0 mode, which can be taken as the field π or the spin-0 longitudinal component of v^a. For details see [39]. The instability of the Fierz-Pauli choice of the mass terms would occur at the quantum level, but as explained in [40], the corrections would occur at a cut-off energy, where the ghost mode would start

to propagate. At energies much lower than the cut-off scale, the corrections could be ignored, and the system is well behaved. We conclude that the proper way of formulating a consistent theory for massive gravity is to adopt the gauge principle for gravitation with extended symmetry and to use the Higgs mechanism to generate mass by breaking the symmetry spontaneously to $SL(2, \mathbb{C})$.

4 Gravity with Hermitian metric

In his attempt to unify gravity with electromagnetism, Einstein proposed to consider a Hermitian metric satisfying the property [17], [18]

$$g_{\mu\nu}(x) = G_{\mu\nu}(x) + iB_{\mu\nu}(x),$$
$$g_{\mu\nu}^{\dagger}(x) = g_{\nu\mu}(x),$$

which implies the symmetries

$$G_{\mu\nu}(x) = G_{\nu\mu}(x), \qquad B_{\mu\nu}(x) = -B_{\nu\mu}(x).$$

In this picture the metric tensor of space-time $G_{\mu\nu}(x)$ is unified geometrically with the field strength $B_{\mu\nu}(x)$ of the electromagnetic field. There is some arbitrariness in the geometric construction due to the non-uniqueness of the connection and a certain choice was taken to remove this ambiguity.

The gauge formulation of gravity with a complex metric has the advantage that it removes the above ambiguity and is very elegant. Assume that we start with the $U(1, 3)$ gauge fields $\omega_{\mu\ b}^{a}$ [41] The $U(1, 3)$ group of transformations is defined as the set of matrix transformations leaving the quadratic form

$$(Z^{a})^{\dagger} \eta_{b}^{a} Z^{b},$$

invariant, where Z^{a} are 4 complex fields and

$$\eta_{b}^{a} = diag(-1, 1, 1, 1),$$

with 3 positive entries. The gauge fields $\omega_{\mu\ b}^{a}$ must then satisfy the condition

$$\left(\omega_{\mu\ b}^{a}\right)^{\dagger} = -\eta_{c}^{b} \omega_{\mu\ d}^{c} \eta_{a}^{d}.$$

The curvature associated with this gauge field is

$$R_{\mu\nu\ b}^{a} = \partial_{\mu}\omega_{\nu\ b}^{a} - \partial_{\nu}\omega_{\mu\ b}^{a} + \omega_{\mu\ c}^{a}\omega_{\nu\ b}^{c} - \omega_{\nu\ c}^{a}\omega_{\mu\ b}^{c}.$$

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Under gauge transformations we have

$$^g\omega_{\mu\ b}^{\ a} = M_c^a \omega_{\mu\ d}^{\ c} M_b^{-1d} - M_c^a \partial_\mu M_b^{-1c},$$

where the matrices M are subject to the condition:

$$(M_c^a)^\dagger \eta_b^a M_d^b = \eta_d^c.$$

The curvature then transforms as

$$^g R_{\mu\nu\ b}^{\ \ a} = M_c^a R_{\mu\nu\ d}^{\ \ c} M_b^{-1d}.$$

Next we introduce the complex vierbein e_μ^a and its inverse e_a^μ defined by

$$e_a^\nu e_\mu^a = \delta_\mu^\nu, \quad e_\nu^a e_b^\nu = \delta_b^a,$$

which transform as

$$^g e_\mu^a = M_b^a e_\mu^b, \quad ^g e_a^\mu = \widetilde{e}_b^\mu M_a^{-1b}.$$

It is also useful to define the complex conjugates

$$e_{\mu a} \equiv \left(e_\mu^a \right)^\dagger, \quad e^{\mu a} \equiv \left(e_a^\mu \right)^\dagger.$$

With this, it is not difficult to see that

$$e_a^\mu R_{\mu\nu\ b}^{\ \ a} \eta_c^b e^{\nu c},$$

is hermitian and $U(1,3)$ invariant. The metric is defined by

$$g_{\mu\nu} = \left(e_\mu^a \right)^\dagger \eta_b^a e_\nu^b,$$

satisfy the property $g_{\mu\nu}^\dagger = g_{\nu\mu}$. When the metric is decomposed into its real and imaginary parts:

$$g_{\mu\nu} = G_{\mu\nu} + i B_{\mu\nu},$$

the hermiticity property then implies the symmetries

$$G_{\mu\nu} = G_{\nu\mu}, \quad B_{\mu\nu} = -B_{\nu\mu}.$$

The gauge invariant Hermitian action is given by

$$I = \int d^4x \sqrt{e}^\dagger e_a^\mu R_{\mu\nu\ b}^{\ \ a} \eta_c^b e^{\nu c} \sqrt{e},$$

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where $e = \det\left(e_\mu^a\right)$. One goes to the second order formalism by integrating out the spin connection and substituting its value in terms of the vierbein. The resulting action depends only on the fields $g_{\mu\nu}$. It is worthwhile to stress that the above action, unlike others proposed to describe nonsymmetric gravity [19] is unique, except for the measure, and unambiguous. Similar ideas have been proposed in the past based on gauging the groups $O(4,4)$ [42],[43] and $GL(4)$ [44], in relation to string duality, but the results obtained there are different from what is presented here.

The infinitesimal gauge transformations for e_μ^a is $\delta e_\mu^a = \Lambda^a_{\ b} e_\mu^b$, which can be decomposed into real and imaginary parts by writing

$$e_\mu^a = e_{0\mu}^a + i e_{1\mu}^a,$$
$$\Lambda^a_{\ b} = \Lambda^a_{0\ b} + i \Lambda^a_{1\ b}.$$

From the gauge transformations of $e_{0\mu}^a$ and $e_{1\mu}^a$ one can easily show that the gauge parameters $\Lambda^a_{0\ b}$ and $\Lambda^a_{1\ b}$ can be chosen to make $e_{0\mu a}$ symmetric in μ and a and $e_{1\mu a}$ antisymmetric in μ and a. This is equivalent to the statement that the Lagrangian should be completely expressible in terms of $G_{\mu\nu}$ and $B_{\mu\nu}$ only, after eliminating $\omega^a_{\mu\ b}$ through its equations of motion. In reality we have

$$G_{\mu\nu} = e_{0\mu}^a e_{0\nu}^b \eta_{ab} + e_{1\mu}^a e_{1\nu}^b \eta_{ab},$$
$$B_{\mu\nu} = e_{0\mu}^a e_{1\nu}^b \eta_{ab} - e_{1\mu}^a e_{0\nu}^b \eta_{ab}.$$

In this special gauge, where we define

$$g_{0\mu\nu} = e_{0\mu}^a e_{0\nu}^b \eta_{ab}, \quad g_{0\mu\nu} g_0^{\nu\lambda} = \delta_\mu^\lambda,$$

and use $e_{0\mu}^a$ to raise and lower indices, we get

$$B_{\mu\nu} = -2 e_{1\mu\nu},$$
$$G_{\mu\nu} = g_{0\mu\nu} - \frac{1}{4} B_{\mu\kappa} B_{\lambda\nu} g_0^{\kappa\lambda}.$$

The last formula appears in the metric of the effective action in open string theory [45].

We can express the Lagrangian in terms of e_μ^a only by solving the $\omega^a_{\mu\ b}$ equations of motion

$$e_a^\mu e^{\nu b} \omega^c_{\nu\ b} + e_b^\nu e^{\mu c} \omega^b_{\nu\ a} - e^{\mu b} e_a^\nu \omega^c_{\nu\ b} - e_b^\mu e^{\nu c} \omega^b_{\nu\ a} =$$
$$\frac{1}{\sqrt{G}} \partial_\nu \left(\sqrt{G} \left(e_a^\nu e^{\mu c} - e_a^\mu e^{\nu c} \right) \right) \equiv X^{\mu c}_{\ \ a}$$

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where $X^{\mu c}{}_a$ satisfy $(X^{\mu c}{}_a)^\dagger = -X^{\mu a}{}_c$. One has to be very careful in working with a nonsymmetric metric

$$g_{\mu\nu} = e^a_\mu e_{\nu a}, \quad g^{\mu\nu} = e^{\mu a} e_{\nu a},$$
$$g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu, \quad g_{\mu\nu} g^{\mu\rho} \neq \delta^\rho_\mu.$$

Care also should be taken when raising and lowering indices with the metric.

Before solving the ω equations, we point out that the trace part of $\omega^a_{\mu b}$ (corresponding to the $U(1)$ part in $U(1,3)$) must decouple from the other gauge fields. It is thus undetermined and decouples from the Lagrangian after substituting its equation of motion. It imposes a condition on the e^a_μ

$$\frac{1}{\sqrt{G}} \partial_\nu \left(\sqrt{G} \left(e^\nu_a e^{\mu a} - e^\mu_a e^{\nu a} \right) \right) \equiv X^{\mu a}{}_a = 0.$$

We can therefore assume, without any loss in generality, that $\omega^a_{\mu b}$ is traceless $\left(\omega^a_{\mu a} = 0 \right)$.

The ω-equation gives

$$\omega_{\kappa\rho}{}^\mu + \omega_\rho{}^\mu{}_\kappa = \frac{1}{8} \delta^\mu_\kappa \left(3 X^\mu_{\rho\mu} - X^\mu_{\mu\rho} \right) + \frac{1}{8} \delta^\mu_\rho \left(-X^\mu_{\kappa\mu} + 3 X^\mu_{\mu\kappa} \right) - X^\mu_{\rho\kappa} \equiv Y^\mu_{\rho\kappa}.$$

We can rewrite this equation after contracting with $e_{\mu c} e^c_\sigma$ to get

$$\omega_{\kappa\rho\sigma} + e^\mu_a e_{\mu c} e^c_\sigma \omega_\rho{}^a{}_\kappa = g_{\sigma\mu} Y^\mu_{\rho\kappa} \equiv Y_{\sigma\rho\kappa}.$$

By writing $\omega_\rho{}^a{}_\kappa = \omega_{\rho\nu\kappa} e^{\nu a}$ we get

$$\left(\delta^\alpha_\kappa \delta^\beta_\rho \delta^\gamma_\sigma + g^{\beta\mu} g_{\sigma\mu} \delta^\alpha_\rho \delta^\gamma_\kappa \right) \omega_{\alpha\beta\gamma} = Y_{\sigma\rho\kappa}.$$

To solve this equation we have to invert the tensor

$$M^{\alpha\beta\gamma}_{\kappa\rho\sigma} = \delta^\alpha_\kappa \delta^\beta_\rho \delta^\gamma_\sigma + g^{\beta\mu} g_{\sigma\mu} \delta^\alpha_\rho \delta^\gamma_\kappa.$$

In the conventional case when all fields are real, the metric $g_{\mu\nu}$ is symmetric and $g^{\beta\mu} g_{\sigma\mu} = \delta^\beta_\sigma$ so that the inverse of $M^{\alpha\beta\gamma}_{\kappa\rho\sigma}$ is simple. In the present case, because of the nonsymmetry of $g_{\mu\nu}$ this is fairly complicated and could only be solved by a perturbative expansion. Writing $g_{\mu\nu} = G_{\mu\nu} + i B_{\mu\nu}$, and defining $G^{\mu\nu} G_{\nu\rho} = \delta^\mu_\rho$ implies that

$$g^{\mu\alpha} g_{\nu\alpha} \equiv \delta^\mu_\nu + L^\mu_\nu,$$
$$L^\mu_\nu = i G^{\mu\rho} B_{\rho\nu} - 2 G^{\mu\rho} B_{\rho\sigma} G^{\sigma\alpha} B_{\alpha\nu} + O(B^3).$$

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The inverse of $M_{\kappa\rho\sigma}^{\alpha\beta\gamma}$ defined by

$$N_{\alpha\beta\gamma}^{\sigma\rho\kappa}M_{\sigma\rho\kappa}^{\alpha'\beta'\gamma'} = \delta_\alpha^{\alpha'}\delta_\beta^{\beta'}\delta_\gamma^{\gamma'}$$

is evaluated to give

$$N_{\alpha\beta\gamma}^{\sigma\rho\kappa} = \frac{1}{2}\left(\delta_\gamma^\sigma\delta_\beta^\rho\delta_\alpha^\kappa + \delta_\beta^\sigma\delta_\alpha^\rho\delta_\gamma^\kappa - \delta_\alpha^\sigma\delta_\gamma^\rho\delta_\beta^\kappa\right)$$
$$- \frac{1}{4}\left(\delta_\beta^\kappa\delta_\alpha^\sigma L_\gamma^\rho + \delta_\alpha^\kappa\delta_\gamma^\sigma L_\beta^\rho - \delta_\gamma^\kappa\delta_\beta^\sigma L_\alpha^\rho\right)$$
$$+ \frac{1}{4}\left(L_\gamma^\kappa\delta_\beta^\sigma\delta_\alpha^\rho + L_\beta^\kappa\delta_\alpha^\sigma\delta_\gamma^\rho - L_\alpha^\kappa\delta_\gamma^\sigma\delta_\beta^\rho\right)$$
$$- \frac{1}{4}\left(\delta_\alpha^\kappa L_\gamma^\sigma\delta_\beta^\rho + \delta_\gamma^\kappa L_\beta^\sigma\delta_\alpha^\rho - \delta_\beta^\kappa L_\alpha^\sigma\delta_\gamma^\rho\right) + O(L^2).$$

This enables us to write

$$\omega_{\alpha\beta\gamma} = N_{\alpha\beta\gamma}^{\sigma\rho\kappa}Y_{\rho\sigma\kappa}.$$

It is clear that the leading term reproduces the Einstein-Hilbert action plus contributions proportional to $B_{\mu\nu}$ and higher order terms. We can check that in the flat approximation for gravity with $G_{\mu\nu}$ taken to be $\delta_{\mu\nu}$, the $B_{\mu\nu}$ field gets the correct kinetic terms. First we write

$$e_\mu^a = \delta_\mu^a - \frac{i}{2}B_{\mu a}, \quad e_{\mu a} = \delta_\mu^a + \frac{i}{2}B_{\mu a}.$$

The $\omega_\mu{}^a{}_a$ equation implies the constraint

$$X^{\mu a}{}_a = \partial_\nu\left(e_a^\mu e^{\nu a} - e_a^\nu e^{\mu a}\right) = 0.$$

This gives the gauge fixing condition $\partial^\nu B_{\mu\nu} = 0$. This equation gave the motivation to Einstein to interpret $B_{\mu\nu}$ as the electromagnetic field strength. We then evaluate

$$\omega_{\mu\nu\rho} = -\frac{i}{2}\left(\partial_\mu B_{\nu\rho} + \partial_\nu B_{\mu\rho}\right).$$

When the $\omega_{\mu\nu\rho}$ is substituted back into the Lagrangian, and after integration by parts one gets

$$L = \omega_{\mu\nu\rho}\omega^{\nu\rho\mu} - \omega_\mu{}^{\mu\rho}\omega_{\nu\rho}{}^\nu = -\frac{1}{4}B_{\mu\nu}\partial^2 B^{\mu\nu}.$$

This is identical to the usual expression $\frac{1}{12}H_{\mu\nu\rho}H^{\mu\nu\rho}$, where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. The later developments of nonsymmetric gravity showed that

the occurrence of the trace part of the spin-connection in a linear form would result in the propagation of ghosts in the field $B_{\mu\nu}$ [20]. This can be traced to the fact that there is no gauge symmetry associated with the field $B_{\mu\nu}$. The inconsistency is avoided by adding to the action the gauge invariant cosmological term [21]

$$\int d^4 x \left(\det e_{\mu a} \det e_\mu^a \right)^{\frac{1}{2}}$$

as this provides a mass term to the field $B_{\mu\nu}$.

5 Unifying space-time and internal symmetries

In D-branes, coordinates of space-time become noncommuting and $U(N)$ matrix-valued [46]

$$\left[X^i, X^j \right] \neq 0.$$

A metric on such spaces will also become matrix-valued. For example in the case of D-0 branes a matrix model action takes the form [47]

$$Tr \left(G_{ij} \left(X \right) \partial_0 X^i \partial_0 X^j \right).$$

At very short distances coordinates of space-time can become noncommuting and represented by matrices.

Developing differential geometry on such spaces is ambiguous. Defining covariant derivatives, affine connections, contracting indices, will all depend on the order these operations are performed because of noncommutativity. Some of these developments lead to inconsistencies such as the occurrence of higher spin fields [48]. In many cases studies were limited to abelian (commuting) matrices with Fierz-Pauli interactions [49]. More recently the spectral approach was taken by Avramidi [50] which implies a well defined order for geometric constructs. Experimentally [51], there is only one massless graviton. Therefore in a consistent $U(N)$ matrix-valued gravity only one massless field should result with all others corresponding to massive gravitons. The masses of the gravitons should be acquired through the Higgs mechanism.

The lesson we learned in the last section is that one should start with a large symmetry and break it spontaneously. The minimal non-trivial extension of $SL(2, \mathbb{C})$ and $U(N)$ is $SL(2N, \mathbb{C})$. This is a non-compact group. It

can be taken as a gauge group only in the first order formalism, in analogy with $SL(2,\mathbb{C})$. The vierbein e_μ^a and the spin-connection ω_μ^{ab} are conjugate variables related by the zero torsion condition. The number of conditions in $T_{\mu\nu}{}^a = 0$ is equal to the number of independent components of ω_μ^{ab}, which can be determined completely in terms of e_μ^a. The $SL(2N,\mathbb{C})$ gauge field can be expanded in the Dirac basis in the form [22],[52]

$$A_\mu = ia_\mu + \gamma_5 b_\mu + \frac{i}{4}\omega_\mu^{ab}\sigma_{ab},$$

where

$$a_\mu = a_\mu^I \lambda^I, \qquad b_\mu = b_\mu^I \lambda^I, \qquad I = 1, \cdots, N^2 - 1,$$
$$\omega_\mu^{ab} = \omega_\mu^{abi}\lambda^i, \qquad i = 0, I.$$

and λ^i are the $U(N)$ Gell-Mann matrices. The analogue of $e_\mu^a \gamma_a$ is

$$L_\mu = e_\mu^a \gamma_a + f_\mu^a \gamma_5 \gamma_a,$$

where e_μ^a and f_μ^a are $U(N)$ matrices. This is equivalent to having complex matrix gravity. The zero torsion condition

$$T = dL + LA + AL = 0,$$

will give two sets of conditions

$$T_{\mu\nu}{}^a = 0, \qquad T_{\mu\nu}{}^{a5} = 0,$$

which will overdetermined the variables ω_μ^{ab}.

The correct approach [52] is to consider $SL(2N,\mathbb{C})\times SL(2N,\mathbb{C})$, or equivalently the complex extension of $SL(2N,\mathbb{C})$ as was done by Isham, Salam and Strathdee [22] for the massive spin-2 nonets. In this case

$$a_\mu = a_\mu^1 + ia_\mu^2, \qquad b_\mu = b_\mu^1 + ib_\mu^2, \qquad \omega_\mu^{ab} = B_\mu^{ab} + iC_\mu^{ab},$$

and the torsion zero constraints are enough to determine B_μ^{ab} and C_μ^{ab} in terms of e_μ^a, f_μ^a, a_μ and b_μ. One can write, almost uniquely, a metric independent gauge invariant action which will correspond to massless $U(N)$ gravitons

$$\int_M Tr\left(i\left(\alpha + \beta\gamma_5\right)LL^{'}F + i\left(\overline{\alpha} + \overline{\beta}\gamma_5\right)L^{'}L\overline{F} + \left(i\lambda + \gamma_5\eta\right)LL^{'}LL^{'}\right),$$

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where $L^{'}$ is related to L. For illustration, the form of this action in the $N = 1$ case is

$$-\frac{1}{2}\int_{M} d^4x \epsilon^{\mu\nu\kappa\lambda} \left(\left(\left(\alpha_2 - \beta_1\right) e_{\mu a}e_{\nu b} + \frac{1}{2}\left(\alpha_1 + \beta_2\right) \epsilon_{abcd}e_{\mu}^{c}e_{\nu}^{d} \right) B_{\kappa\lambda}^{ab} \right.$$

$$+ \left(\left(\alpha_2 + \beta_1\right) f_{\mu a}f_{\nu b} - \frac{1}{2}\left(\alpha_1 - \beta_2\right) \epsilon_{abcd}f_{\mu}^{c}f_{\nu}^{d} \right) C_{\kappa\lambda}^{ab}$$

$$\left. + \epsilon_{abcd}\left((\lambda - \eta) e_{\mu}^{a}e_{\nu}^{b}e_{\kappa}^{c}e_{\lambda}^{d} + (\lambda + \eta) f_{\mu}^{a}f_{\nu}^{b}f_{\kappa}^{c}f_{\lambda}^{d} \right) \right).$$

where $B_{\kappa\lambda}^{ab}$ $C_{\kappa\lambda}^{ab}$ are the curvatures associated with B_{μ}^{ab} and C_{μ}^{ab} respectively. To give masses to the spin-2 fields, introduce the Higgs fields H and $H^{'}$ transforming as L and $L^{'}$ and constrained in such a way as to break the symmetry non-linearly from $SL(2N,\mathbb{C}) \times SL(2N,\mathbb{C})$ to $SL(2,\mathbb{C})$ [38]. We can add the mass terms

$$\int_{M} Tr \left((i\tau + \gamma_5 \xi) LH^{'} LH^{'} LL^{'} + (i\rho + \gamma_5 \xi) HL^{'} HL^{'} LL^{'} \right).$$

Some of the relevant terms in the quadratic parts of the action are, in component form [52],

$$\int d^4x \epsilon^{\mu\nu\kappa\lambda} Tr \left(\alpha_1 \left\{ E_{\mu}^{a}, E_{\nu a}^{'} \right\} a_{\kappa\lambda}^{2} + \alpha_2 \left\{ F_{\mu}^{a}, F_{\nu a}^{'} \right\} b_{\kappa\lambda}^{2} \right)$$

$$+ \epsilon_{abcd}Tr \left(\beta_1 \left\{ E_{\mu}^{a}, E_{\nu}^{'b} \right\} B_{\kappa\lambda}^{cd} + \beta_2 \left\{ F_{\mu}^{a}, F_{\nu}^{'b} \right\} C_{\kappa\lambda}^{cd} + \right.$$

$$\left. + \gamma_1 E_{\mu}^{a}E_{\nu}^{'b}E_{\kappa}^{c}E_{\lambda}^{'d} + \gamma_2 F_{\mu}^{a}F_{\nu}^{'b}F_{\kappa}^{c}F_{\lambda}^{'d} + \delta_1 E_{\mu}^{a}E_{\nu}^{'b}E_{\kappa}^{c}F_{\lambda}^{d} + \delta_2 F_{\mu}^{a}F_{\nu}^{'b}F_{\kappa}^{c}E_{\lambda}^{d} \right).$$

This action is complicated because all expressions are matrix valued. Equations are solved perturbatively. The action can be determined to second order in the fields, and the spectrum found to be given by two sets of $SU(N)$ matrix-valued massive gravitons, plus two singlets of gravitons, one massless and the other is massive, as well as $SU(N) \times SU(N)$ gauge fields.

We decompose $E_{\mu a}^{I}$ into symmetric and antisymmetric parts

$$E_{\mu a}^{I} = S_{\mu a}^{I} + T_{\mu a}^{I},$$

where $S_{\mu a}^{I} = S_{a\mu}^{I}$ is symmetric and $T_{\mu a}^{I} = -T_{a\mu}^{I}$ is antisymmetric. The symmetric part propagates while the antisymmetric part $T_{\mu\nu}^{I}$ couples to the

Yang-Mills fields and act as auxiliary fields to give them kinetic energies. For example, besides the quadratic terms for $T^{\mu\nu I}$ coming from the mass terms, we have

$$\int\limits_M d^4x \left(\partial_\mu a_\nu^{1I} - \partial_\nu a_\mu^{1I} \right) T^{\mu\nu I},$$

as well as similar couplings to a_μ^{2I}, b_μ^{1I}, b_μ^{2I}. By eliminating the field $T_{\mu\nu}^I$ the fields a_μ^{1I}, a_μ^{2I}, b_μ^{1I}, b_μ^{2I} would acquire the regular $SU(N)$ Yang-Mills gauge field strengths. A detailed study of this system is carried in [52]. Again, this shows the effectiveness of the gauge principle in generalizing unambiguosly the metric and gravitational interactions to become matrix valued. This is to be contrasted with the geometric approach, which is plagued with ambiguities.

6   Supergravity from gauging graded Lie algebras

The gauge formulation of gravity is simpler than the geometrical formulation. However, the simplifications are much more apparent in the derivation of the supergravity action based on gauging the supersymmetry algebra. In 1976 the supergravity action was constructed by extending the local supersymmetric invariance to the Einstein action by using the Noether's method [24]. This is a perturbative approach of insuring the invariance order by order, up to quartic fermionic terms and is very complicated. It was natural to attempt construct this theory using the gauge approach, and indeed this was done soon after and resulted in the most elegant formulation of supergravity. The starting point is to consider the supersymmetry algebra. This is the graded extension of the Poincaré algebra by adding the following commutation relations [25]

$$[S_\alpha, J_{ab}] = \frac{i}{2} \left(\gamma_{ab}\right)_\alpha^\beta S_\beta,$$
$$[S_\alpha, P_a] = 0,$$
$$\{S_\alpha, S_\beta\} = - \left(\gamma^a C\right)_{\alpha\beta} P_a,$$

where P_a, J_{ab} and S_α are, respectively, the translation, rotation and fermionic generators and $C_{\alpha\beta}$ is the charge conjugation matrix. Demanding invariance

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under local supersymmetry transformations, requires introducing the covariant derivative

$$D_\mu = \partial_\mu + e_\mu^a P_a + \omega_\mu^{ab} J_{ab} + \overline{\psi}_\mu^\alpha S_\alpha$$

where e_μ^a, ω_μ^{ab} and $\overline{\psi}_\mu^\alpha$ are now gauge fields with the following fermionic transformations

$$\delta e_\mu^a = \overline{\eta}\gamma^a \psi_\mu$$
$$\delta \omega_\mu^{ab} = 0,$$
$$\delta\overline{\psi}_\mu^\alpha = \left[\overline{\eta} \left(\overleftarrow{\partial}_\mu - \frac{1}{4}\gamma_{ab}\omega_\mu^{ab} \right) \right]^\alpha.$$

The fermionic gauge parameters η are dependent on the coordinates x^μ. The field strengths are found by computing the commutator

$$[D_\mu, D_\nu] = C_{\mu\nu}{}^a P_a + \frac{1}{4}R_{\mu\nu}{}^{ab}\gamma_{ab} + \overline{D}_{\mu\nu}^\alpha S_\alpha,$$

where

$$C_{\mu\nu}{}^a = \partial_\mu e_\nu^a + \omega_\mu^{ab} e_{\nu b} - \frac{1}{2}\overline{\psi}_\mu \gamma^a \psi_\nu - \mu \leftrightarrow \nu,$$
$$R_{\mu\nu}{}^{ab} = \partial_\mu \omega_\nu^{ab} + \omega_\mu^{ac}\omega_{\mu c}{}^b - \mu \leftrightarrow \nu,$$
$$\overline{D}_{\mu\nu}^\alpha = \overline{\psi}_{[\nu} \left(\overleftarrow{\partial}_{\mu]} - \frac{1}{4}\gamma_{ab}\omega_{\mu]}^{ab} \right)^\alpha - \mu \leftrightarrow \nu.$$

The field strengths transform covariantly under supersymmetry transformations

$$\delta C_{\mu\nu}{}^a = \overline{\eta}\gamma^a \psi_{\mu\nu},$$
$$\delta R_{\mu\nu}{}^{ab} = 0,$$
$$\delta\overline{\psi}_{\mu\nu}^\alpha = \frac{1}{4}\left(\overline{\eta}\gamma_{ab}R_{\mu\nu}{}^{ab} \right)^\alpha$$

In analogy with the bosonic case, we assume that the generalized torsion $C_{\mu\nu}{}^a$, which is the field strength associated with translational symmetry, to vanish. It is then necessary to modify the transformations of ω_μ^{ab} in order to maintain this condition, thus we must take

$$\delta' \omega_{\mu ab} = -\frac{1}{4}\left(\overline{\eta}\gamma_a D_{\mu b} - \overline{\eta}\gamma_b D_{\mu a} + e_\mu^c e_a^\rho e_b^\sigma \overline{\eta}\gamma_c D_{\rho\sigma} \right).$$

The action is then constructed as function of the remaining field strengths:

$$I = \frac{1}{4}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}\left(\epsilon_{abcd}e_\mu^a e_\nu^b R_{\rho\sigma}^{\ \ cd} + i\alpha e_\mu^a \overline{\psi}_\nu \gamma_5 \gamma_a D_{\rho\sigma}\right)$$

where the parameter α is determined by the requirement that the $\omega_\mu^{\ ab}$ equation of motion, which appears linearly and quadratically, should result in the equation $C_{\mu\nu}^{\ \ a} = 0$. This fixes the parameter $\alpha = 1$. Using the observation that the variation of the action with respect to $\omega_\mu^{\ ab}$ drops out because

$$\left.\frac{\delta I}{\delta\omega_\mu^{\ ab}}\right|_{e,\psi} = 0,$$

proving the invariance of the action under supersymmetry transformations become an easy matter. Varying the Einstein term only gives one term

$$2\epsilon_{abcd}\overline{\eta}\gamma^a\psi_\mu e_\nu^b R_{\rho\sigma}^{\ \ cd},$$

while varying the fermionic term results in three terms

$$i\overline{\eta}\gamma^a\psi_\mu\overline{\psi}_\nu\gamma_5\gamma_a D_{\rho\sigma} + ie_\mu^a\overline{\eta}\overleftarrow{D}_\nu\gamma_5\gamma_a D_{\rho\sigma} - \frac{i}{4}e_\mu^a\overline{\psi}_\nu\gamma_5\gamma_a\gamma_{cd}\eta R_{\rho\sigma}^{\ \ cd}.$$

Integrating by parts the middle term gives two terms, one proportional to $D_{[\mu}e_{\nu]}^a$ which can be equated to $\frac{1}{2}\overline{\psi}_\mu\gamma^a\psi_\nu$ using the generalized torsion condition, while the other is proportional to $D_{[\nu}D_{\rho\sigma]}$ which by the Bianchi identity is equal to $\frac{1}{4}R_{[\nu\rho}^{\ \ cd}\gamma_{cd}\psi_{\sigma]}$. One can show by a simple Fierz reshuffle that these two terms cancel the first and third terms in the above expression. The simplicity of this derivation [25] should be contrasted with the extreme complexity of the Noether's method which was originally used to derive this Lagrangian [24].

Another method of deriving the supergravity action is based on the observation that the supersymmetry algebra could be obtained by an Inonü-Wigner contraction of the Orthosymplectic algebra $OSP(4,1)$ [26],[27],[53]. The group is defined as the set of linear transformations which leave invariant the bilinear form

$$(z,z) = z^A \eta_{AB} z^B,$$

where the linear space $\{z^A = \theta^\alpha, z\}$ comprise one commuting coordinate $z^5 = z$ and four anticommuting coordinates $z^A = \theta^\alpha$, $A = 1, \cdots, 4$. The matrix η_{AB} is given by

$$\eta_{AB} = \begin{pmatrix} C_{\alpha\beta} & 0 \\ 0 & 1 \end{pmatrix},$$

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where $C_{\alpha\beta}$ is an antisymmetric root of the unit matrix [27]. Indices are raised and lowered with η_{AB} and its inverse η^{AB}, where $\eta^{AB}\eta_{BC} = \delta^A_C$, thus

$$z_A = \eta_{AB}z^B, \quad z^A = \eta^{AB}z_B.$$

The tensor ϕ^B_A, a representation of $OSP(1,4)$, transforms like $z_A z^B$, contains three distinct irreducible representations. These are the graded trace $(-1)^a \phi^A_A$, the graded symmetric and traceless part, and the graded antisymmetric part. The grading number a for z^A is defined by

$$a = 0, \quad A = 5$$
$$a = 1, \quad A = 1, \cdots, 4$$

so that $z^A z^B = (-1)^{ab} z^B z^A$ and the graded symmetric and antisymmetric tensors are defined by

$$\phi^{(s)}_{AB} = (-1)^{ab} \phi^{(s)}_{BA},$$
$$\phi^{(a)}_{AB} = -(-1)^{ab} \phi^{(a)}_{BA},$$
$$(-1)^a \phi^{(s)\,A}_A = 0,$$

where $\phi_{AB} = \phi^C_A \eta_{CB}$.

The matrix decomposition of these representations are given by [27]

$$\phi^{(s)B}_A = \begin{pmatrix} \left(\frac{1}{4}\varphi + \gamma\pi + \gamma_a\gamma_5 v^a\right)^\beta_\alpha & \lambda_\alpha \\ -\overline{\lambda}^\alpha & \phi \end{pmatrix},$$

$$\phi^{(a)B}_A = \begin{pmatrix} \left(\gamma_a\phi^a + \frac{1}{4}\gamma_{ab}\phi^{ab}\right)^\beta_\alpha & \lambda_\alpha \\ \overline{\lambda}^\alpha & 0 \end{pmatrix},$$

where the spinors used are Majorana, $\lambda_\alpha = C_{\alpha\beta}\overline{\lambda}^\beta$. The antisymmetric representation is the adjoint representation. Demanding local $OSP(1,4)$ gauge invariance is done along similar lines to the pure bosonic case. First introduce gauge potentials in the adjoint representation

$$\Phi_\mu = \begin{pmatrix} \left(\kappa^{-1}(i\gamma_a)^\beta_\alpha e^a_\mu + \frac{1}{4}(\gamma_{ab})^\beta_\alpha \omega^{ab}_\mu\right) & \kappa^{-\frac{1}{2}}\psi_{\mu\alpha} \\ \kappa^{-\frac{1}{2}}\overline{\psi}^\alpha_\mu & 0 \end{pmatrix}$$

which transforms according to

$$\Phi_\mu = \Omega\Phi_\mu\Omega^{-1} + \Omega\partial_\mu\Omega^{-1}$$

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where Ω are gauge parameters in the adjoint representation. The field strengths of the gauge potentials Φ_μ are defined by

$$\Phi_{\mu\nu} = \partial_\mu\Phi_\nu - \partial_\nu\Phi_\mu + [\Phi_\mu, \Phi_\nu]$$
$$= \begin{pmatrix} \left(\kappa^{-1}(i\gamma_a)_\alpha^\beta C_{\mu\nu}{}^a + \frac{1}{4}(\gamma_{ab})_\alpha^\beta F_{\mu\nu}{}^{ab}\right) & \kappa^{-\frac{1}{2}}\chi_{\mu\nu\alpha} \\ \kappa^{-\frac{1}{2}}\overline{\chi}_{\mu\nu}^\alpha & 0 \end{pmatrix}$$

where

$$C_{\mu\nu}{}^a = \partial_\mu e_\nu^a + \omega_\mu{}^{ab}e_{\nu b} - \frac{i}{4}\overline{\psi}_\mu\gamma^a\psi_\nu - \mu \leftrightarrow \nu,$$
$$F_{\mu\nu}{}^{ab} = R_{\mu\nu}{}^{ab} - 4\kappa^{-2}\left(e_\mu^a e_\nu^b - e_\mu^a e_\nu^b\right) - \kappa^{-1}\overline{\psi}_\mu\gamma^{ab}\psi_\nu,$$
$$\chi_{\mu\nu}^\alpha = D_{\mu\nu}^\alpha + 2\kappa^{-1}(i\gamma_a)_\beta^\alpha e_{[\mu}^a \psi_{\nu]b}^\beta.$$

The orthosymplectic algebra $OSP(4,1)$ contracts to the supersymmetry algebra by taking the limit $\kappa \to \infty$. This corresponds to the contraction of a de Sitter space with infinite radius to give Minkowski space. The component gauge fields transform according to

$$\delta e_\mu^a = \partial_\mu\varepsilon^a + \omega_\mu{}^{ab}\varepsilon_b + \overline{\eta}\gamma^a\psi_\mu + \omega^{ab}e_{\mu b}$$
$$\delta\omega_\mu^{ab} = \partial_\mu\omega^{ab} + 2\omega^{[ac}\omega_{\mu c}{}^{b]} + 2\kappa^{-2}e_\mu^{[a}\varepsilon^{b]} + 2\kappa^{-1}\overline{\eta}\gamma^{ab}\psi_\mu,$$
$$\delta\psi_\mu = \partial_\mu\eta + \frac{1}{4}\omega_\mu{}^{ab}\gamma_{ab}\eta + \frac{1}{4}\omega^{ab}\gamma_{ab}\psi_\mu + i\kappa^{-1}e_\mu^a\gamma_a\eta + i\kappa^{-1}\varepsilon^a\gamma_a\psi_\mu$$

Imposing the constraint on the field strength of the translation generators

$$C_{\mu\nu}{}^a = 0$$

requires the modification of the transformations of ω_μ^{ab} in order to maintain this constraint. The invariant action is then formed from the other non-vanishing components of $\Phi_{\mu\nu}$, and is given by

$$I = \int d^4x \epsilon^{\mu\nu\rho\sigma}\left(\epsilon_{abcd}F_{\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd} + \alpha\overline{\chi}_{\mu\nu}\gamma_5\chi_{\rho\sigma}\right),$$

where the parameter α is fixed by gauge invariance to be

$$\alpha = \frac{i}{2}.$$

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The above action splits into five parts ranging in powers of κ from κ^0 to κ^{-4}. The first is the Gauss-Bonnet topological invariant

$$I^{(0)} = \frac{1}{4} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}{}^{ab} R_{\rho\sigma}{}^{cd}.$$

The second part is a boundary term and is given by

$$I^{(-1)} = i\kappa^{-1} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \overline{D}_{\mu\nu} \gamma_5 D_{\rho\sigma}.$$

The third part is the supergravity action

$$I^{(-2)} = 4\kappa^{-2} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \left(\epsilon_{abcd} e^a_\mu e^b_\nu R_{\rho\sigma}{}^{cd} + i e^a_\mu \overline{\psi}_\nu \gamma_5 D_{\rho\sigma} \right).$$

The fourth part part is a mass-like term for the gravitino

$$I^{(-2)} = 4i\kappa^{-3} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} e^a_\mu e^b_\nu \overline{\psi}_\rho \gamma_5 \psi_\sigma,$$

and the last part is a cosmological term

$$I^{(-2)} = 16\kappa^{-4} \int d^4 x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} e^a_\mu e^b_\nu e^c_\rho e^d_\sigma.$$

The total Lagrangian is that of supergravity with a cosmological constant, mass like term for the gravitino and boundary terms. Using the fact that the first two terms are boundary terms and can be discarded, we notice that in the limit $\kappa \to \infty$, the action reduces to the supergravity action $I^{(-2)}$ which is invariant under the supersymmetry algebra. This is to be expected because the supersymmetry algebra is obtained from the orthosymplectic algebra $OSP(4,1)$ by an Inonü-Wigner contraction by taking the limit $\kappa \to \infty$. This analysis shows the power of the gauge idea, and reduce an intractable calculation to the evaluation of a simple trace. This was just the starting point for gauging graded Lie groups, and many applications of this idea followed soon after. This idea played an important role in the construction of extended supergravities [54] and conformal supergravity [29].

7   Topological gravity in odd dimensions

It was shown by Witten [55] that the Einstein-Hilbert action with or without a cosmological term for three dimensional gravity can be derived in the first

order formalism as a gauge theory of $SO(1,4)$, $SO(2,3)$, or $ISO(1,3)$. The action is of the Chern-Simons type and is renormalizable. This formalism could be generalized to all odd dimensions. Consider the $2n+1$ Chern-Simons form ω_{2n+1} defined by [56]

$$\omega_{2n+1} = (n+1) \int\limits_0^1 \delta t \left\langle A \left(t dA + t^2 A^2 \right)^n \right\rangle,$$

where A is the gauge field for one of the gauge groups $ISO(1,2n)$, $SO(1,2n+1)$ or $SO(2,2n)$ depending on whether we want to gauge the Poincaré de Sitter or anti de Sitter groups [57],[58] in $2n+1$ dimensions. In the definition of the bracket $\langle \cdots \rangle$ it is essential to use the $(n+1)$ group-invariant form [55]

$$\left\langle J_{A_1 B_1} J_{A_2 B_2} \cdots J_{A_n B_n} \right\rangle = \epsilon_{A_1 B_1 A_2 B_2 \cdots A_n B_n},$$

where J_{AB} is the group generator and $A, B = 0, 1, \cdots, 2n+1$. The action is taken to be

$$I_{2n+1} = k \int\limits_{M^{2n+1}} \omega_{2n+1}.$$

Under a gauge transformation the gauge field A transforms according to

$$A^g = g^{-1} A g + g^{-1} dg$$

which implies that the Chern-Simons form transforms to

$$\omega_{2n+1}^g = \omega_{2n+1} + d\alpha_{2n} + (-1)^n \frac{(n)!\,(n+1)!}{(2n+1)!} \left\langle \left(g^{-1} dg \right)^{2n+1} \right\rangle,$$

where α_{2n} is a two form which is a function of A and $g^{-1} dg$, and the last term is proportional to the winding number. For the groups $ISO(1,2n)$, $SO(1,2n+1)$ or $SO(2,2n)$ the winding number is proportional to torsion, and thus vanishes. For example in the case of five-dimensional Chern-Simons form the homotopy elements are

$$\pi_5\left(SO(1,5)\right) = \pi_5\left(SO(5)\right) = Z_2,$$
$$\pi_5\left(SO(2,4)\right) = \pi_5\left(SO(4)\right) = Z_2 + Z_2.$$

Thus for manifolds without boundary the Chern-Simons action is gauge invariant, which implies that the constant k is not quantized, which is desirable

if the theory is to describe gravity. For manifolds with boundary the action is invariant provided that A or $g^{-1}dg$ vanish at the boundary. We shall assume that the manifold M_{2n+1} is without boundary.

To make the connection to gravity we identify

$$A^{ab} = \omega^{ab}, \quad A^{a,2n+1} = e^a, \quad a = 0, 1, \cdots, 2n.$$

The remarkable thing is that the above Chern-Simons action when expressed in terms of the fields e^a and ω^{ab} takes the form

$$I_{2n+1} = \int_{M_{2n+1}} \sum_{l=0}^{n} \frac{1}{2l+1} \lambda^l \binom{n}{l} \cdot$$
$$\epsilon_{a_1 a_2 \cdots a_{2n+1}} R^{a_1 a_2} \wedge \cdots R^{a_{2n-2l-1} a_{2n-2l}} \wedge e^{a_{2n-2l+1}} \wedge \cdots \wedge e^{a_{2n+1}},$$

where

$$\lambda = \begin{cases} 1 & \text{for } SO(2, 2n) \\ -1 & \text{for } SO(1, 2n+1) \\ 0 & \text{for } ISO(1, 2n) \end{cases} .$$

Thus the Chern-Simons action in odd dimensions is seen to be the sum of Euler densities with fixed coefficients [59]. Notice that in the $ISO(1, 2n)$ case only one term in the action remains

$$I_{2n+1} = \int_{M_{2n+1}} \epsilon_{a_1 a_2 \cdots a_{2n+1}} R^{a_1 a_2} \wedge \cdots R^{a_{2n-1} a_{2n}} \wedge e^{a_{2n+1}}.$$

The variational equations of A^{aB} have a simple structure

$$\epsilon_{A_1 B_1 \cdots A_n B_n} F^{A_1 B_1} \wedge \cdots \wedge F^{A_n B_n} = 0$$

where

$$F^{AB} = dA^{AB} + A^{AC} \wedge A_C{}^B.$$

Decomposing this equation with respect to the new variables gives two equations

$$\epsilon_{a_1 a_2 \cdots a_{2n+1}} \left(R^{a_1 a_2} + \lambda e^{a_1} e^{a_2} \right) \wedge \cdots \wedge \left(R^{a_{2n-1} a_{2n}} + \lambda e^{a_{2n-1}} e^{a_{2n}} \right) = 0,$$
$$\epsilon_{a_1 a_2 \cdots a_{2n+1}} T^{a_1} \wedge \left(R^{a_2 a_3} + \lambda e^{a_2} e^{a_3} \right) \wedge \cdots \wedge \left(R^{a_{2n-2} a_{2n-1}} + \lambda e^{a_{2n-2}} e^{a_{2n-1}} \right) = 0,$$

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where T^a is the torsion given by

$$T^a = de^a + \omega^{ab} \wedge e_b.$$

This scheme could be generalized to include fermions by gauging the graded extensions of the Poincaré and de-Sitter groups. The supersymmetric extension of the Poincaré group is known to exist in all dimensions, but those extending the de Sitter groups are limited. The list stops at the supersymmetric extension of $O(2,6)$ corresponding to a seven-dimensional space-time in this framework. The supergroups relevant to space-time dimensions of four or more are [60]

$$\left\{ \begin{array}{c} (O(2,3) \oplus O(N), (4,N)), \quad N = 1,2,\cdots \\ \left(O(1,4) \oplus U(1), \left(4 + \overline{4}\right)\right) \end{array} \right\} \qquad D = 4$$

$$\left\{ \begin{array}{c} \left(O(2,3) \oplus O(N), (4,N) + \left(\overline{4}, \overline{N}\right)\right), \quad N = 1,2,\cdots \\ \left(O(2,4) \oplus SU(4), (4+4) + (\overline{4}, \overline{4})\right) \end{array} \right\} \qquad D = 5$$

$$\left\{ \begin{array}{c} (O(1,6) \oplus SU(2), (8,2)) \\ (O(2,5) \oplus SU(2), (8,2)) \end{array} \right\} \qquad D = 6$$

$$(O(2,6) \oplus SU(N,q), (8,2N)), \quad N = 1,2,\cdots \qquad D = 7$$

In this notation, the first part in the parentheses gives the bosonic parts of the group while the second part specifies the fermionic representations with respect to these bosonic groups. The supergravity actions are constructed using the Chern-Simons forms in odd dimensions with the gauge fields taking values in the graded Lie algebras corresponding to that dimension. The graded group invariant is now constructed using the supertrace. The super-Poincaré groups are easier to deal with and a general formula can be given. In dimensions higher than seven, the smallest extensions of the de Sitter groups are the orthosymplectic groups, which have many more generators than the minimum required. There are many further developments in this direction [61], for a review see [62].We therefore see that the gauge principle is very easy and powerful in determining topological theories of gravity in odd dimension. This is to be contrasted with the geometrical approach where such constructions are quite difficult.

8 Noncommutative gravity

Open string theories as well as D-branes in the presence of a background antisymmetric B-field give rise to noncommutative effective field theories [46],[45]. This is equivalent to field theories deformed with the star product [63]. The primary example of this is noncommutative $U(N)$ Yang-Mills theory. It is natural then to ask whether it is possible to deform Einstein's gravity with the star product. This is not easy to do in a geometrical setting, although there has been some recent progress [64]. On the other hand this is possible in the gauge approach using the same methods used before. To construct a noncommutative gravitational action in four dimensions one proceeds as follows. First the gauge field strength of the noncommutative gauge group $SO(4,1)$ is taken. This is followed by an Inonü-Wigner contraction to the group $ISO(3,1)$, thus determining the dependence of the deformed vierbein on the undeformed one.

The starting point is the assumption that space-time coordinates x^μ do not commute

$$[x^\mu,\, x^\nu] = i\theta^{\mu\nu}$$

where $\theta^{\mu\nu}$ are assumed to be constant. However, under diffeomorphism transformations, $\theta^{\mu\nu}$ becomes a function of x, and one has to generalize the definition of the star product to be applicable for a general manifold, but this is only known for symplectic manifolds [65]. The effect of this noncommutativity is that ordinary products are replaced with the star product defined by

$$f * g = e^{\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\xi^\mu}\frac{\partial}{\partial\eta^\nu}} f\left(x+\xi\right) g\left(x+\eta\right)|_{\xi=\eta=0}.$$

In gauge theories one mainly uses $U(N)$ gauge fields subject to the condition $\widehat{A}_\mu^\dagger = -\widehat{A}_\mu$ because such condition could be maintained under the gauge transformations [45]

$$\widehat{A}_\mu^g = \widehat{g} * \widehat{A}_\mu * \widehat{g}_*^{-1} - \widehat{g} * \partial_\mu \widehat{g}_*^{-1},$$

where $\widehat{g} * \widehat{g}_*^{-1} = 1 = \widehat{g}_*^{-1} * \widehat{g}$. To avoid using complex or Hermitian gravitational fields, we introduce the gauge fields $\widehat{\omega}_\mu^{AB}$ [66] subject to the conditions [67],[68]

$$\widehat{\omega}_\mu^{AB\dagger}\left(x,\theta\right) = -\widehat{\omega}_\mu^{BA}\left(x,\theta\right),$$
$$\widehat{\omega}_\mu^{AB}\left(x,\theta\right)^r \equiv \widehat{\omega}_\mu^{AB}\left(x,-\theta\right) = -\widehat{\omega}_\mu^{BA}\left(x,\theta\right).$$

Expanding the gauge fields in powers of θ, we have

$$\widehat{\omega}_\mu{}^{AB}(x,\theta) = \omega_\mu{}^{AB} - i\theta^{\nu\rho}\omega_{\mu\nu\rho}{}^{AB} + \cdots.$$

The above conditions then imply the following

$$\omega_\mu{}^{AB} = -\omega_\mu{}^{BA}, \qquad \omega_{\mu\nu\rho}{}^{AB} = \omega_{\mu\nu\rho}{}^{BA}.$$

A basic assumption to be made is that there are no new degrees of freedom introduced by the new fields, and that these are related to the undeformed fields by the Seiberg-Witten map [45]. This is defined by the property

$$\widehat{\omega}_\mu{}^{AB}(\omega) + \delta_{\widehat{\lambda}}\widehat{\omega}_\mu{}^{AB}(\omega) = \widehat{\omega}_\mu{}^{AB}(\omega + \delta_\lambda\omega),$$

where $\widehat{g} = e^{\widehat{\lambda}}$ and the infinitesimal transformation of $\omega_\mu{}^{AB}$ is given by

$$\delta_\lambda\omega_\mu{}^{AB} = \partial_\mu\lambda^{AB} + \omega_\mu{}^{AC}\lambda^{CB} - \lambda^{AC}\omega_{\mu C}{}^{B},$$

and for the deformed field it is

$$\delta_{\widehat{\lambda}}\widehat{\omega}_\mu{}^{AB} = \partial_\mu\widehat{\lambda}^{AB} + \widehat{\omega}_\mu{}^{AC} * \widehat{\lambda}^{CB} - \widehat{\lambda}^{AC} * \widehat{\omega}_\mu{}^{CB}.$$

To solve this equation we first write

$$\widehat{\omega}_\mu{}^{AB} = \omega_\mu{}^{AB} + \omega'_\mu{}^{AB}(\omega)$$
$$\widehat{\lambda}^{AB} = \lambda^{AB} + \lambda'^{AB}(\lambda,\omega)$$

where $\omega'_\mu{}^{AB}(\omega)$ and $\lambda'^{AB}(\lambda,\omega)$ are functions of θ, and then substitute into the variational equation to get [45]

$$\omega'_\mu{}^{AB}(\omega + \delta\omega) - \omega'_\mu{}^{AB}(\omega)$$
$$= \partial_\mu\lambda'^{AB} + \omega_\mu{}^{AC}\lambda'^{B}_C - \lambda'^{AC}\omega_{\mu C}{}^{B} + \omega'_\mu{}^{AC}\lambda^{B}_C - \lambda^{AC}\omega'_{\mu C}{}^{B}$$
$$+ \frac{i}{2}\theta^{\nu\rho}\left(\partial_\nu\omega_\mu{}^{AC}\partial_\rho\lambda^{B}_C + \partial_\rho\lambda^{AC}\partial_\nu\omega_{\mu C}{}^{B}\right)$$

This equation is solved, to first order in θ, by

$$\widehat{\omega}_\mu{}^{AB} = \omega_\mu{}^{AB} - \frac{i}{4}\theta^{\nu\rho}\{\omega_\nu, \partial_\rho\omega_\mu + R_{\rho\mu}\}^{AB} + O(\theta^2)$$
$$\widehat{\lambda}^{AB} = \lambda^{AB} + \frac{i}{4}\theta^{\nu\rho}\{\partial_\nu\lambda, \omega_\rho\}^{AB} + O(\theta^2)$$

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where we have defined the anticommutator $\{\alpha,\ \beta\}^{AB} \equiv \alpha^{AC}\beta_C^{\ B} + \beta^{AC}\alpha_C^{\ B}$. With this it is possible to derive the differential equation that govern the dependence of the deformed fields on θ to all orders

$$\delta\widehat{\omega}_\mu^{\ AB}(\theta) = -\frac{i}{4}\theta^{\nu\rho}\left\{\widehat{\omega}_{\nu,*}\ \partial_\rho\widehat{\omega}_\mu + \widehat{R}_{\rho\mu}\right\}^{AB}$$

with the products in the anticommutator given by the star product, and where

$$\widehat{R}_{\mu\nu}^{\ \ AB} = \partial_\mu\widehat{\omega}_\nu^{\ AB} - \partial_\nu\widehat{\omega}_\mu^{\ AB} + \widehat{\omega}_\mu^{\ AC} * \widehat{\omega}_{\nu C}^{\ \ B} - \widehat{\omega}_\nu^{\ AC} * \widehat{\omega}_\mu^{\ CB}$$

We are mainly interested in determining $\widehat{\omega}_\mu^{\ AB}(\theta)$ to second order in θ. This is due to the fact that the deformed gravitational action is required to be hermitian. The undefomed fields being real, then implies that all odd powers of θ in the action must vanish. The above equation could be solved iteratively, by inserting the solution to first order in θ in the differential equation and integrating it. The second order corrections in θ to $\widehat{\omega}_\mu^{\ AB}$ are

$$\frac{1}{32}\theta^{\nu\rho}\theta^{\kappa\sigma}\left(\{\omega_\kappa,\ 2\{R_{\sigma\nu},\ R_{\mu\rho}\} - \{\omega_\nu,(D_\rho R_{\sigma\mu} + \partial_\rho R_{\sigma\mu})\} - \partial_\sigma\{\omega_\nu,\ (\partial_\rho\omega_\mu + R_{\rho\mu})\}\}^{AB}\right.$$
$$\left. + [\partial_\nu\omega_\kappa,\ \partial_\rho(\partial_\sigma\omega_\mu + R_{\sigma\mu})]^{AB} - \{\{\omega_\nu,\ (\partial_\rho\omega_\kappa + R_{\rho\kappa})\},\ (\partial_\sigma\omega_\mu + R_{\sigma\mu})\}^{AB}\right)$$

One problem remains of how to determine the dependence of the vierbein $\widehat{e}_\mu^a$ on the undeformed field as it is not a gauge field. To resolve this problem we adopt the strategy of considering the field e_μ^a as the gauge field of the translation generator of the inhomegenious Lorentz group, obtained through the contraction of the group $SO(4,1)$ to $ISO(3,1)$. We write $\widehat{\omega}_\mu^{\ a5} = k\widehat{e}_\mu^a$ and $\widehat{\omega}_\mu^{\ 55} = k\widehat{\phi}_\mu$. We shall only impose the condition $T_{\mu\nu}^{\ \ a} = 0$ and not $\widehat{T}_{\mu\nu}^{\ \ a} = 0$ because we are not interested in ϕ_μ which will drop out in the limit $k \to 0$. The result for $\widehat{e}_\mu^a$ in the limit $k \to 0$ is

$$\widehat{e}_\mu^a = e_\mu^a - \frac{i}{4}\theta^{\nu\rho}\left(\omega_\nu^{\ ac}\partial_\rho e_{\mu c} + \left(\partial_\rho\omega_\mu^{\ ac} + R_{\rho\mu}^{\ \ ac}\right)e_{\nu c}\right)$$
$$+ \frac{1}{32}\theta^{\nu\rho}\theta^{\kappa\sigma}\left(2\{R_{\sigma\nu},\ R_{\mu\rho}\}^{ac}\ e_{\kappa c} - \omega_\kappa^{\ ac}\left(D_\rho R_{\sigma\mu}^{\ \ cd} + \partial_\rho R_{\sigma\mu}^{\ \ cd}\right)e_{\nu d}\right.$$
$$- \{\omega_\nu,(D_\rho R_{\sigma\mu} + \partial_\rho R_{\sigma\mu})\}^{ad}\ e_{\kappa d} - \partial_\sigma\{\omega_\nu,\ (\partial_\rho\omega_\mu + R_{\rho\mu})\}^{ac}\ e_{\kappa c}$$
$$- \omega_\kappa^{\ ac}\partial_\sigma\left(\omega_{\nu c}^{\ \ d}\partial_\rho e_{\mu d} + \left(\partial_\rho\omega_\mu^{\ cd} + R_{\rho\mu}^{\ \ cd}\right)e_{\nu d}\right) + \partial_\nu\omega_\kappa^{\ ac}\partial_\rho\partial_\sigma e_{\mu c}$$
$$- \partial_\rho\left(\partial_\sigma\omega_\mu^{\ ac} + R_{\sigma\mu}^{\ \ ac}\right)\partial_\nu e_{\kappa c} - \{\omega_\nu,\ (\partial_\rho\omega_\kappa + R_{\rho\kappa})\}^{ac}\ \partial_\sigma e_{\mu c}$$
$$\left. - \left(\partial_\sigma\omega_\mu^{\ ac} + R_{\sigma\mu}^{\ \ ac}\right)\left(\omega_{\nu c}^{\ \ d}\partial_\rho e_{\kappa d} + \left(\partial_\rho\omega_{\kappa c}^{\ \ d} + R_{\rho\kappa}^{\ \ cd}\right)e_{\nu d}\right)\right) + O\left(\theta^3\right)$$

At this point, it is possible to determine the deformed curvature and use it to calculate the deformed action given by [66]

$$\int d^4x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abce} \widehat{e}^a_\mu * \widehat{R}_{\nu\rho}{}^{bc} * \widehat{e}^d_\sigma$$

Of course the actual expression obtained after substituting for the fields $e^\mu_{a\rho\tau}$, $e^\mu_{a\rho\tau\kappa\sigma}$, $\omega^{ab}_{\mu\rho\tau}$ and $\omega^{ab}_{\mu\rho\tau\kappa\sigma}$ is very complicated, and it is not clear whether one can associate a geometric structure with it. One can, however, take this expression and study the deformations to the graviton propagator, which will receive θ^2 corrections.

9 Conclusions

The simple idea that started with Utiyama to formulate the general theory of relativity as a gauge theory of the Lorentz group has grown to become a powerful tool in investigating gravitational theories. We have sampled only few of the known applications in the literature, the full extent of which is considerable. The main advantages are the simplicity and straightforwardness of the formalism. The idea can be applied to shed light on the various aspects of gravity. We have shown that a consistent formulation of massive gravity is possible through the use of spontaneous breakdown of gauge symmetry. When applied to graded Lie algebras it gives supergravity, and for the Chern-Simons action it gives topological gravity. By extending the gauge algebra to become complex, one obtains complex gravity with a Hermitian metric. It is also possible to give a consistent deformation of the Einstein action on spaces where ordinary products are replaced with star products. All this work gives the promise that the gauge principle can unify gravity with the other fundamental interactions, all of which are known to be based on the gauge theories.

10 Acknowledgment

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